Journal of Statistical Physics, Vol. 114, Nos. 5/6, March 2004 (© 2004)

A Counter-Example to the Theorem of Hiemer and Snurnikov

Thierry Monteil[1]

Received July 7, 2003; accepted October 23, 2003

A planar polygonal billiard $\mathscr{P}$ is said to have the finite blocking property if for every pair (O, A) of points in $\mathscr{P}$ there exists a finite number of "blocking" points $B_1,..., B_n$ such that every billiard trajectory from O to A meets one of the B_i's. As a counter-example to a theorem of Hiemer and Snurnikov, we construct a family of rational billiards that lack the finite blocking property.

KEY WORDS: Rational polygonal billiards; translation surfaces; blocking property.

1. INTRODUCTION

A planar polygonal billiard $\mathscr{P}$ is said to have the finite blocking property if for every pair (O, A) of points in $\mathscr{P}$ there exists a finite number of "blocking" points $B_1,..., B_n$ (different from O and A) such that every billiard trajectory from O to A meets one of the B_i's.

In ref. 1, Hiemer and Snurnikov tried to prove that any rational polygonal billiard has the finite blocking property. The aim of this paper is to construct a family of rational billiards that lack the finite blocking property.

2. THE COUNTER-EXAMPLE

Let α be a positive irrational number and $\mathscr{P}_\alpha$ be the polygon drawn in Fig. 1 (L_1 and L_2 can be chosen arbitrarily, greater than 1).

[1] Institut de Mathématiques de Luminy, CNRS UPR 9016, Case 907, 163 Avenue de Luminy, 13288 Marseille cedex 09, France; e-mail: monteil@iml.univ-mrs.fr

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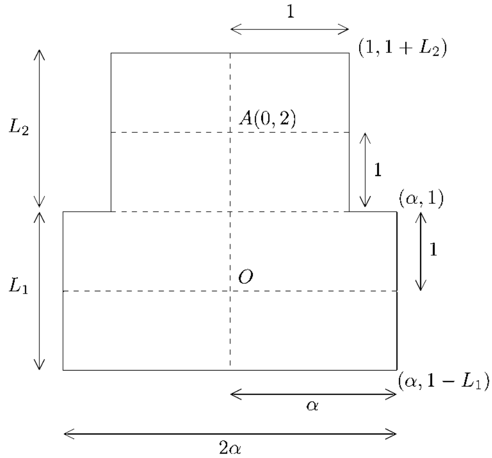

Fig. 1. The polygon $\mathscr{P}_\alpha$.

Let $(p_n, q_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{N}^{*2}$ such that:

- q_n is strictly increasing
- $|p_n - q_n\alpha| < 1$.

For example, we can take $q_n = n+1$ and $p_n = [q_n\alpha]$.

For $n \in \mathbb{N}$, let γ_n be the billiard trajectory starting from O to A with slope

$$\frac{1}{p_n + q_n\alpha} = \frac{1}{2q_n\alpha + \lambda_n} = \frac{1}{2p_n - \lambda_n}$$

where $\lambda_n = p_n - q_n\alpha \in \,]-1, 1[$.

So, we can check (with the classical unfolding procedure shown in Fig. 2) that γ_n hits q_n walls, passes through $(\lambda_n, 1)$, hits p_n walls and then passes through $A(0, 2)$.

The fact that $\lambda_n \in \,]-1, 1[$ enables us to avoid the banana peel shown in Fig. 3.

Now, we assume by contradiction that there is a point $B(x, y)$ in $\mathscr{P}_\alpha$ distinct from O and A such that infinitely many γ_n pass through B. Hence, there is a subsequence such that for all n in $\mathbb{N}$, γ_{i_n} passes through B.

There are two cases to consider:

First case: $y \in \,]0, 1]$. By looking at the unfolded version of the trajectory (Fig. 2), we see that $x = \varepsilon_{i_n} y(p_{i_n} + q_{i_n}\alpha)$ [mod 2α] where $\varepsilon_{i_n} \in \{-1, 1\}$ depends on the parity of the number of bounces of γ_{i_n} from O to B.

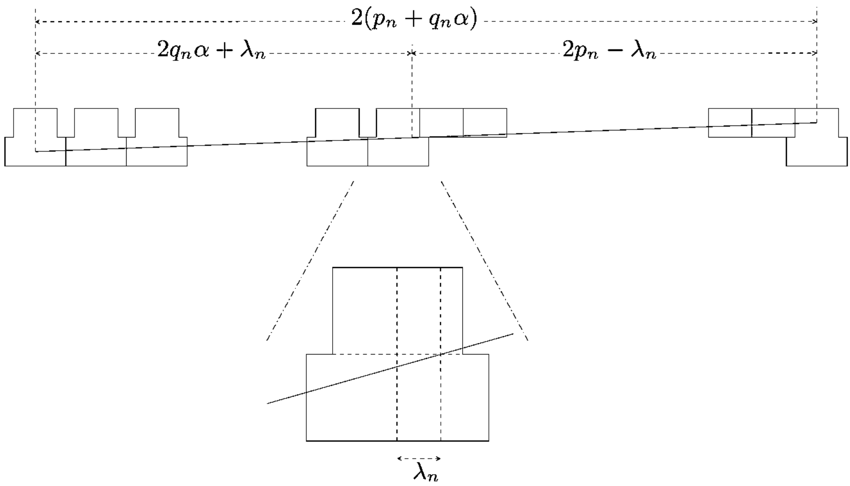

Fig. 2. The unfolding procedure.

So, there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$ such that $x = \varepsilon_{i_n} y(p_{i_n} + q_{i_n}\alpha) + 2k_{i_n}\alpha$.

Taking a further subsequence, we can consider $\varepsilon \circ i$ to be constant with value ε.

We have $x = \varepsilon y(p_{i_0} + q_{i_0}\alpha) + 2k_{i_0}\alpha = \varepsilon y(p_{i_1} + q_{i_1}\alpha) + 2k_{i_1}\alpha$.

Hence, $(p_{i_1} - p_{i_0}) + (q_{i_1} - q_{i_0})\,\alpha = \frac{\varepsilon 2\alpha}{y}\,(k_{i_0} - k_{i_1}) \neq 0$.

So, $\frac{\varepsilon 2\alpha}{y}$ can be written as $r + s\alpha$ where r and s are rational numbers.

Now, if $n \geqslant 1$, we still have $(p_{i_n} - p_{i_0}) + (q_{i_n} - q_{i_0})\,\alpha = (r + s\alpha)(k_{i_0} - k_{i_n})$.

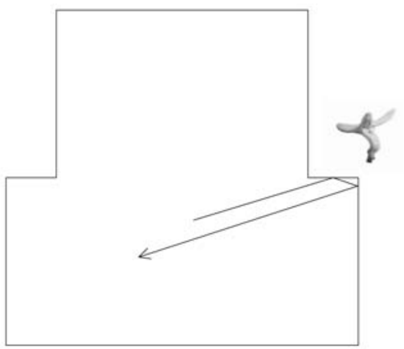

Fig. 3. The banana peel.

Because $(1, \alpha)$ is free over $\mathbb{Q}$, we have

- $(p_{i_n} - p_{i_0}) = r(k_{i_0} - k_{i_n})$
- $(q_{i_n} - q_{i_0}) = s(k_{i_0} - k_{i_n}) \neq 0$ (remember that q_n is strictly increasing).

Thus, by dividing,

$$\frac{r}{s} = \frac{p_{i_n} - p_{i_0}}{q_{i_n} - q_{i_0}} = \frac{p_{i_n}}{q_{i_n}} \left(1 - \frac{p_{i_0}}{p_{i_n}}\right)\left(\frac{1}{1 - \frac{q_{i_0}}{q_{i_n}}}\right) \xrightarrow[n\to\infty]{} \alpha \in \mathbb{R} \setminus \mathbb{Q}$$

leading to a contradiction.

For the second case, if $y \in [1, 2[$, it is exactly the same (take the point $A(0, 2)$ as the origin and reverse Fig. 2).

Thus, **the billiard $\mathscr{P}_\alpha$ lacks the finite blocking property**.

3. CONCLUSION

In ref. 3, we study Hiemer and Snurnikov's proof: it works for rational billiards with discrete translation group (such billiards are called *almost integrable*). Then we generalize the notion of finite blocking property to translation surfaces (see ref. 2 for precise definitions). With an analogous construction to the one described above, we obtain the following results:

Theorem 1. Let $n \geqslant 3$ be an integer. The following assertions are equivalent:

- the regular n-gon has the finite blocking property.
- the right-angled triangle with an angle equal to π/n has the finite blocking property.
- $n \in \{3, 4, 6\}$.

Theorem 2. A translation surface that admits cylinder decomposition of commensurable moduli in two transversal directions has the finite blocking property if and only if it is a torus branched covering.

Corollary 1. A Veech surface has the finite blocking property if and only if it is a torus branched covering.

Note that torus branched coverings are the analogue (in the vocabulary of translation surfaces) of almost integrable billiards.

We also provide a local sufficient condition for a translation surface to fail the finite blocking property: it enables us to give a complete classification for the L-shaped surfaces and a density result in the space of translation surfaces in every genus $g \geqslant 2$.

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